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Letter to the Editor

A non-standard finite difference scheme for the equations modelling stellar structure

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One of the most important problems in astrophysics is the determination of the internal structure of stars using the fundamental principles of physics [1–3]. An excellent introduction to this topic is the book by Chandrasekhar [1] and the book chapter of Rouse [3]. Within the set of issues of concern is the calculation of both radial and non-radial oscillatory modes generated by the propagation of acoustical energy within the stellar interior [4,5] and their comparison with the observational data [6].

The basic equations for a spherically symmetric star are given below. They correspond to four coupled, non-linear ordinary differential equations [1–4] giving the change, as a function of radial distance r, of the pressure (P), mass (M), luminosity (L), and temperature (T):

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\left(\frac{GM\rho}{r^2}\right), \quad \frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi r^2\rho, \quad \frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2\rho\varepsilon, \tag{1–3}$$

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\left(\frac{\gamma_e - 1}{\gamma_e}\right)\left(\frac{GMT}{r^2}\right), \tag{4}$$

where $\rho(r)$ is the density given by the equation of state for an ideal gas [1],

$$\rho = \frac{P}{KT}, \tag{5}$$

with K being a known, positive constant; ε is the non-negative "energy production function" given in terms of ρ and T, i.e.,

$$\varepsilon(\rho, T) \geqslant 0; \tag{6}$$

G is the Newton gravitational constant [1,3]; and γ_e is a constant greater than one. The physics of this model dictates that certain monotonic and positivity properties hold for the solutions to Eqs. (1)–(4) [1–4]. In particular, all four variables must be non-negative functions of r. Further, the pressure $P(r)$ decreases monotonic from the center of the star to its surface, with the same

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property holding for the temperature $T(r)$. Likewise, the mass $M(r)$ and luminosity $L(r)$ both increase monotonic over the same range in the radial variable r.

A major difficulty arises when Eqs. (1)–(4) are numerically integrated [3,4]. The particular numerical integration scheme used may lead to solutions for which the conditions of positivity and/or monotonicity are violated. Such behaviors are called numerical instabilities and are solutions of the discrete equations that do not correspond to any solutions of the original differential equations. The main purpose of this communication is to show that using the non-standard finite difference techniques of Mickens [7], a discrete scheme can be constructed such that both positivity and monotonicity are maintained. Consequently, none of the elementary numerical instabilities will occur. As will be seen in the calculations to follow, the four dependent variables are to be calculated in the order

$$P \to M \to L \to T. \tag{7}$$

In the remainder of our discussion, the following notation will be used to represent the discretized variables

$$r \to r_m = (\Delta r)m, \quad \Delta r = h, \tag{8}$$

$$[P(r), M(r), L(r), T(r)] \to [P_m, M_m, L_m, T_m], \tag{9}$$

where P_m is an approximation for $P(r_m)$, etc.

First, consider the "P" Eq. (1) which can be rewritten using the ideal gas equation of state to the form:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\left(\frac{G}{K}\right)\left(\frac{MP}{r^2T}\right). \tag{10}$$

The right-side can be expressed as

$$\frac{\mathrm{d}R}{\mathrm{d}r} = -2\left(\frac{G}{K}\right)\left(\frac{MP}{r^2T}\right) + \left(\frac{G}{K}\right)\left(\frac{MP}{r^2T}\right). \tag{11}$$

The following non-standard scheme [7,8] is selected to numerically integrate Eq. (11)

$$\frac{P_{m+1} - P_m}{h} = -2\left(\frac{G}{K}\right)\left(\frac{M_m P_{m+1}}{r_m^2 T_m}\right) + \left(\frac{G}{K}\right)\left(\frac{M_m P_m}{r_m^2 T_m}\right). \tag{12}$$

Note that P_{m+1} appears linearly and solving for it gives

$$P_{m+1} = \left[\frac{r_m^2 T_m + (\frac{hG}{K})M_m}{r_m^2 T_m + 2(\frac{hG}{K})M_m}\right] P_m. \tag{13}$$

Also, observe that if

$$T_m > 0, \quad M_m > 0, \quad P_m > 0, \tag{14}$$

then the scheme of Eq. (13) has the following two properties:

$$P_{m+1} > 0, \quad P_{m+1} < P_m. \tag{15}$$

Consequently, the computed values of pressure satisfy the correct conditions of positivity and monotonicity.

Replacing the density on the right-side of Eq. (2) by the expression in Eq. (5) gives

$$\frac{\mathrm{d}M}{\mathrm{d}r} = \left(\frac{4\pi}{K}\right)\frac{r^2 P}{T}. \tag{16}$$

The corresponding finite difference scheme is [7,8]

$$\frac{M_{m+1} - M_m}{h} = \left(\frac{4\pi}{K}\right)\frac{r_m^2 P_{m+1}}{T_m}, \tag{17}$$

where it should be noted that the pressure is evaluated at r_{m+1} rather than r_m. However, P_{m+1} is already known and is given in terms of (T_m, M_m) by Eq. (13). Solving for M_{m+1} gives

$$M_{m+1} = M_m + \left(\frac{4\pi h}{K}\right)\left(\frac{r_m^2 P_{m+1}}{T_m}\right). \tag{18}$$

Inspection of this last equation shows that

$$M_m > 0, \quad P_m > 0, \quad T_m > 0 \tag{19}$$

implies that

$$M_{m+1} > 0, \quad M_{m+1} > M_m. \tag{20}$$

In other words, the numerical scheme provides solutions with the correct positivity and monotonicity properties.

Eq. (3) contains on its right-side a given "energy production" function that is non-negative, i.e.,

$$\varepsilon(\rho, T) = \varepsilon\left(\frac{P}{KT}, T\right) \geqslant 0, \tag{21}$$

where the density ρ has been replaced by the relation of Eq. (5). To proceed, define new functions $E(\rho, T)$ and $E_1(P, T)$ as follows:

$$\begin{aligned} E(\rho, T) &\equiv \rho\varepsilon(\rho, T) \\ &= \left(\frac{P}{KT}\right)\varepsilon\left(\frac{P}{KT}, T\right) \\ &\equiv E_1(P, T). \end{aligned} \tag{22}$$

With these definitions Eq. (3) can be expressed as

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 E_1(P, T), \tag{23}$$

and the corresponding finite difference scheme is

$$\frac{L_{m+1} - L_m}{h} = 4\pi r_m^2 E_1(P_{m+1}, T_m). \tag{24}$$

Solving for L_{m+1} gives

$$L_{m+1} = L_m + 4\pi r_m^2 E_1(P_{m+1}, T_m), \tag{25}$$

and this form shows directly that the calculated luminosity function is non-negative and increasing in a monotone fashion.

Finally, turning to the "T" Eq. (4), it can be rewritten as

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -2\beta\left(\frac{MT}{r^2}\right) + \beta\left(\frac{MT}{r^2}\right), \tag{26}$$

where

$$\beta = \left(\frac{\gamma_e - 1}{\gamma_e}\right) G > 0. \tag{27}$$

The corresponding finite difference scheme is

$$\frac{T_{m+1} - T_m}{h} = -2\beta\left(\frac{M_{m+1}T_{m+1}}{r_m^2}\right) + \beta\left(\frac{M_{m+1}T_m}{r_m^2}\right), \tag{28}$$

which upon solving for T_{m+1} gives

$$T_{m+1} = \left[\frac{r_m^2 + (h\beta)M_{m+1}}{r_m^2 + 2(h\beta)M_{m+1}}\right] T_m. \tag{29}$$

Inspection of Eq. (29) shows that the calculated temperature function is non-negative and monotonic decreasing.

In summary, a non-standard finite difference scheme has been constructed for the four coupled, non-linear ordinary differential equations modelling the interior structure of stars. This scheme is *dynamically consistent* with the original differential equations, i.e., the numerical solutions satisfy the conditions of positivity and monotonicity, just as the differential equations themselves do. Consequently, the elementary numerical instabilities will not occur and the proposed scheme should provide superior numerical solutions as compared to the use of standard methods [4,5].

The proposed scheme is sequential, i.e., the four dependent variables are numerically computed in a definite order as presented below:

1. From (r_m, M_m, P_m), the value P_{m+1} is calculated; see Eq. (13).
2. Next, M_{m+1} is determined from (P_{m+1}, T_m, M_m); see Eq. (18).
3. The luminosity, L_{m+1}, is then calculated from values of (L_m, T_m, P_{m+1}); see Eq. (25).
4. Finally, T_{m+1} is calculated from (T_m, M_{m+1}); see Eq. (29).

It should also be indicated that the proposed scheme will provide an improved integration procedure for calculating the acoustical modes giving rise to both radial and non-radial oscillations in stellar models [4].

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